Multimodality in the Kalman Filter and Ensemble Kalman Filter

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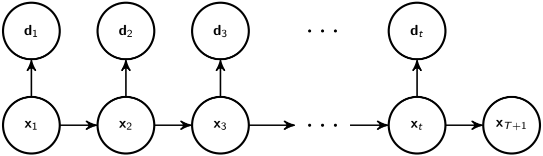
Kalman Model



Process model's assumptions:

1. Gaussian initial distribution $f(\mathbf{x}_1)$
2. Single site dependence and conditional independence
3. Gauss-linear forward and likelihood model:

$$f(\mathbf{x}_{t+1}|\mathbf{x}_t) = \varphi_p(\mathbf{x}_{t+1}, \mathbf{B}\mathbf{x}_t, \mathbf{\Sigma}_{x|x})$$
$$f(\mathbf{d}_t|\mathbf{x}_t) = \varphi_p(\mathbf{d}_t, \mathbf{H}\mathbf{x}_t, \mathbf{\Sigma}_{d|x})$$

[Kalman(1960)],[Myrseth and Omre(2010)]

Kalman Model

Properties

Properties:

1. Analytically tractable, conjugate prior
2. Models linear unimodal processes

Selection Gaussian distribution

Let $\mathbf{A} \subset \mathbb{R}^q$,and

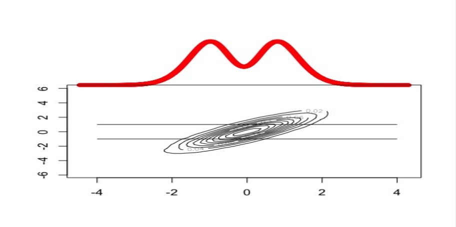
$$\begin{bmatrix} \mathbf{x}_0 \\ \boldsymbol{\nu} \end{bmatrix} \sim \varphi_{p+q}\left(\begin{bmatrix} \mathbf{x}_0 \\ \boldsymbol{\nu} \end{bmatrix}; \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_{x_0} \\ \boldsymbol{\mu}_{\nu} \end{bmatrix}, \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix} \right)$$

then $\mathbf{x}_{0,A} = [\mathbf{x}_0 | \boldsymbol{\nu} \in A]$ is Selection Gauss.

Flexibility

1. Skewness
2. Multimodality
3. Conjugate prior to a Gauss-linear likelihood and forward model

Bimodality



[Azzalini and Valle(1996)],[Rimstad and Omre(2014)]

Selection Gaussian Kalman Model

Model's assumptions:

1. Selection Gaussian initial distribution $f(\mathbf{x}_1)$
2. Single site response and conditional independence
3. Gauss-linear forward and likelihood model:

$$
\begin{aligned}
f(\mathbf{x}_{t+1}|\mathbf{x}_t) =& \varphi_p(\mathbf{x}_{t+1}; \mathbf{B}\mathbf{x}_t, \mathbf{\Sigma}_{x|x}) \\
f(\mathbf{d}_t|\mathbf{x}_t) =& \varphi_p(\mathbf{d}_t; \mathbf{H}\mathbf{x}_t, \mathbf{\Sigma}_{d|x})
\end{aligned}
$$

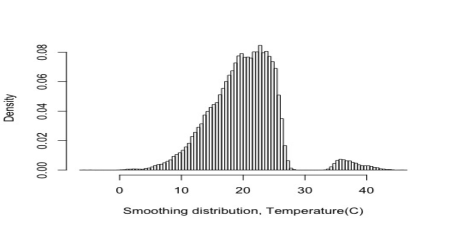
[Naveau et al.(2005)]

Selection Gaussian Kalman Model

Properties

1. Analytically tractable
2. Models multimodality
3. Easy to implement

Marginal smoothing distribution



Implementation

1. We start with $\begin{bmatrix} \mathbf{x}_1 \\ \boldsymbol{\nu} \end{bmatrix}$ that is Gaussian
2. We increment (update) to $\begin{bmatrix} \mathbf{x}_1 \\ \boldsymbol{\nu} \\ \mathbf{d}_1 \end{bmatrix}$ that is still Gaussian
3. We increment (forward) to $\begin{bmatrix} \mathbf{x}_2 \\ \mathbf{x}_1 \\ \boldsymbol{\nu} \\ \mathbf{d}_1 \end{bmatrix}$
4. etc ...

Implementation

1. Access to Kalman filtering $\mathbf{x}_t|\mathbf{d}_1, ..., \mathbf{d}_t$, smoothing $\mathbf{x}_s|\mathbf{d}_1, ..., \mathbf{d}_t, s \leq t$ and inversion $\mathbf{x}_1|\mathbf{d}_1, ..., \mathbf{d}_T$.
2. Fast computation
3. Conserve a Gaussian structure

Example: Backtracking the 2D Heat equation

The heat equation:

$$\frac{\partial T}{\partial t} - \nabla^2 T = 0$$
$$\nabla T . \mathbf{n} = 0$$

Modelled using finite differences on $[0, 1] \times [0, 1]$, it gives the following Gauss-linear forward model:

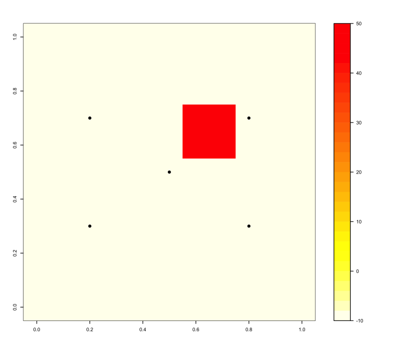
$$f(\mathbf{T}_{t+1}|\mathbf{T}_t) = \varphi_p(\mathbf{T}_{t+1}, \mathbf{B}\mathbf{T}_t, \mathbf{\Sigma}_{T|T}) \quad (1)$$

Data is collected at 5 different locations using the following Gauss-linear likelihood model:

$$f(\mathbf{d}_t|\mathbf{T}_t) = \varphi_p(\mathbf{d}_t, \mathbf{H}\mathbf{T}_t, \mathbf{\Sigma}_{d|T}) \quad (2)$$

Example: Backtracking the 2D Heat equation

Initial Heat map

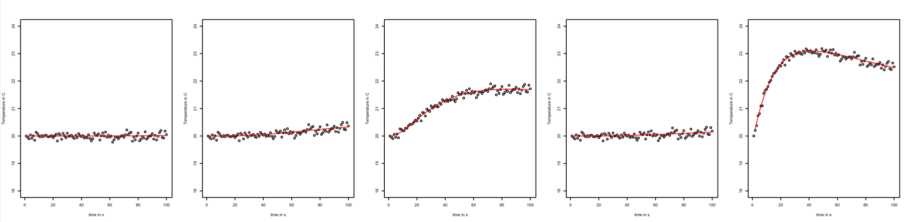


Facts

1. Discontinuous initial conditions
2. 5 data collection points

Example: Backtracking the 2D Heat equation

Data collected Vs True process

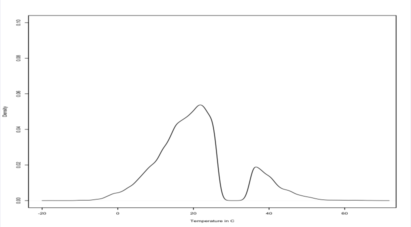


Parameters

1. $dt = 1s$
2. $\mathbf{\Sigma}_{d|x} = 0.01\mathbf{I}$.

Initial model: A reflection of our a priori knowledge

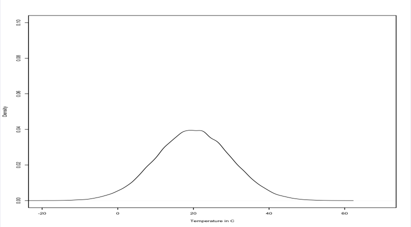
Scenario 1: Sel-Gauss initial model



Properties

1. Two lobes.

Scenario 2:Gaussian initial model



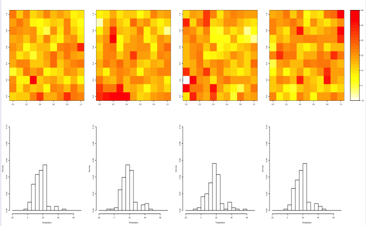
Properties

1. $E(\mathbf{x}_1) = 20.$
2. $\mathbf{Var}(\mathbf{x}_1) = 100.$

Initial model: A reflection of our a priori knowledge

Realizations from the initial distribution:

Sel-Gauss initial distribution



Gaussian initial distribution

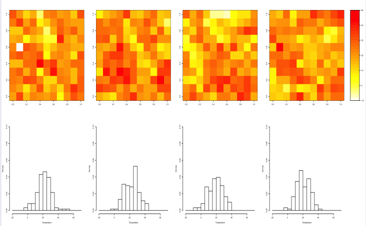
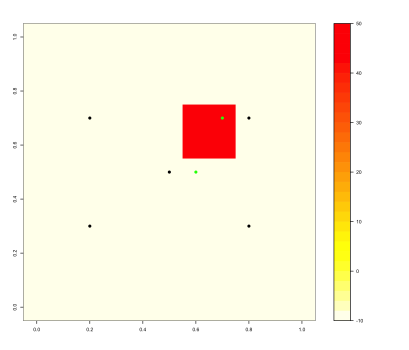


Exhibit $[x_{1,i}|d_1, \ldots, d_T]$ at 2 different locations

Initial Heat map

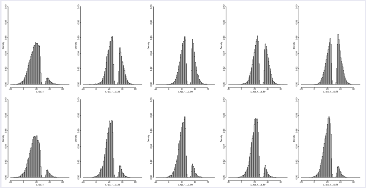


Facts

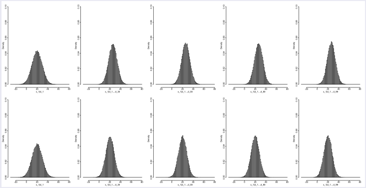
1. Compare the marginal distribution at two different point
2. One inside, one outside

Exhibit $[x_{1,i}|d_1, ..., d_T]$ at 2 different locations

Sel-Gauss initial model



Gaussian initial model

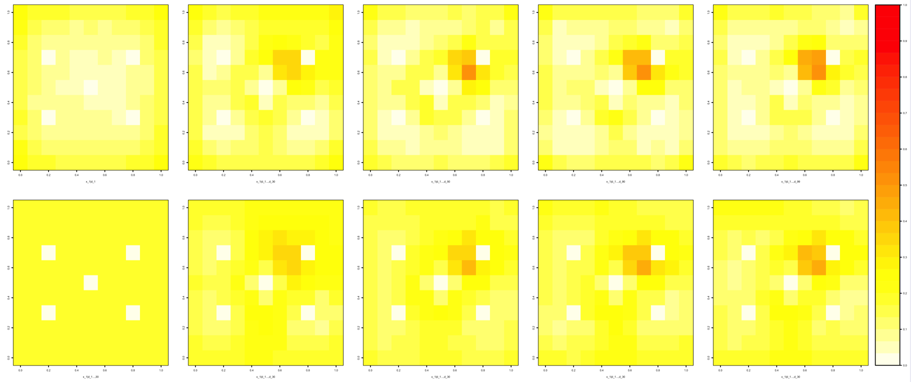


Global behavior

We define $LR(\mathbf{x})$ as:

$$LR_i(\mathbf{x}) = P(x_i > 28,75) \quad \forall i \in [1, p]$$

$LR(\mathbf{x}_1 | d_1, ..., d_t)$ for different values of t



Algorithm:EnKF for the Sel-Gauss (EnKF(SG))

- Initiate

$$n_e = \text{no. of ensemble members}$$
$$\begin{bmatrix} \mathbf{x}_0^{u(i)} \\ \boldsymbol{\nu}_0^{u(i)} \end{bmatrix}, \; i = 1, ..., n_e \text{ iid. } f(\mathbf{x}_0^u, \boldsymbol{\nu}_0^u) = N(\boldsymbol{\mu}_0^u, \boldsymbol{\Sigma}_0^u)$$
$$\mathbf{d}_0^{(i)} = \mathbf{H}\mathbf{x}_0^{u(i)} + \boldsymbol{\eta}_0^i, \; i = 1, ..., n_e \text{ with } \boldsymbol{\eta}_0 \sim N(0, \boldsymbol{\Sigma}_0^{d|x})$$

- Iterate $t = 0, ..., T$

$$\text{Estimate } \boldsymbol{\Sigma}_{x,\nu,d} \text{ from } \{(\mathbf{x}_t^{u(i)}, \boldsymbol{\nu}_t^{u(i)}, \mathbf{d}_t^i), i = 1, ..., n_e\}$$
$$\begin{bmatrix} \mathbf{x}_t^{c(i)} \\ \boldsymbol{\nu}_t^{c(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_t^{u(i)} \\ \boldsymbol{\nu}_t^{u(i)} \end{bmatrix} + \boldsymbol{\Gamma}_{x,\nu,d}\boldsymbol{\Sigma}_d^{-1}(\mathbf{d}_t - \mathbf{d}_t^i), \; i = 1, ..., n_e$$
$$\begin{bmatrix} \mathbf{x}_{t+1}^{u(i)} \\ \boldsymbol{\nu}_{t+1}^{u(i)} \end{bmatrix} = \begin{bmatrix} g(\mathbf{x}_t^{c(i)}) \\ \boldsymbol{\nu}_t^{c(i)} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\delta}_t \\ \mathbf{0} \end{bmatrix}, \; i = 1, ..., n_e \text{ with } \boldsymbol{\delta}_t \sim N(0, \boldsymbol{\Sigma}_t^{x|x})$$
$$d_{t+1}^{(i)} = \mathbf{H}\mathbf{x}_{t+1}^{u(i)} + \boldsymbol{\eta}_{t+1}^i, \; i = 1, ..., n_e \text{ with } \boldsymbol{\eta}_{t+1} \sim N(0, \boldsymbol{\Sigma}_{t+1}^{d|x})$$

- Estimate $\boldsymbol{\mu}_{T+1}^u, \boldsymbol{\Sigma}_{T+1}^u$ and assess $f(\mathbf{x}_{T+1}|\mathbf{d}_0, ..., \mathbf{d}_T, \nu \in A)$

Algorithm:EnKF for the Sel-Gauss (EnKF(SG))

1. Non gaussian output: Ensemble of $\mathbf{x}, \boldsymbol{\nu}$ rather than $\mathbf{x}|\boldsymbol{\nu} \in A$
2. Forward step made easy by :

$$g(\mathbf{x}_t|\nu \in A) = g(\mathbf{x}_t)|\boldsymbol{\nu} \in A$$

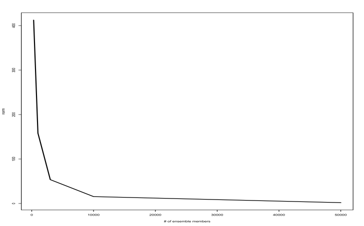
Test on a linear forward model: Previous example

Consider now:

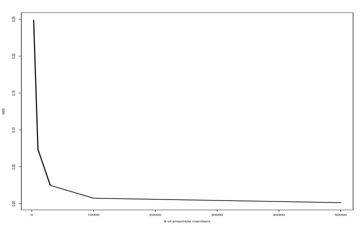
$$\begin{bmatrix} \mathbf{x}_{t+1}^{u(i)} \\ \boldsymbol{\nu}_{t+1}^{u(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{t}^{c(i)} \\ \boldsymbol{\nu}_{t}^{c(i)} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\delta}_t \\ \mathbf{0} \end{bmatrix} \quad i = 1, \dots, n_e$$

We "show" that the EnKF(SG) converges numerically to the Selection Gauss Kalman Filter as $n_e \to \infty$ when the forward model is linear.

Expected Value: Norm of the difference for $x_2|d_1, d_2$



Covariance matrix: Norm of the difference for $x_2|d_1, d_2$



Ongoing work: Use EnKF for parameter estimation

1. Idea: Put a Sel-Gauss prior on the parameter, one lobe per possible value for the parameter (diffusivity coefficient, but also porosity).
2. Use the EnKF to estimate the parameters.

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